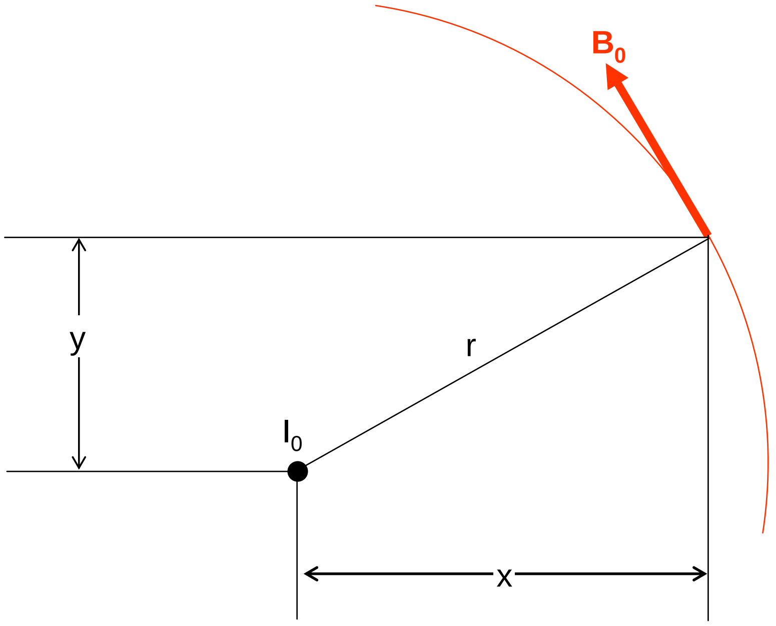
How to calculate the magnetic field from a two-core DC cable

We will consider here a DC circuit, made up two conductors (“go” and “return”, or “+” and “-“), with infinitely long, straight conductors.

To build up the calculation in stages, consider first just a single current, I_0.



The magnetic field it produces forms concentric circles. So at a point (x,y), the magnetic field is in the direction shown as B_0 (we have assumed the current is coming out of the page).

The size of the magnetic field is given by Ampere’s law:

$$B = \frac{\mu_0 I_0}{2\pi r}$$

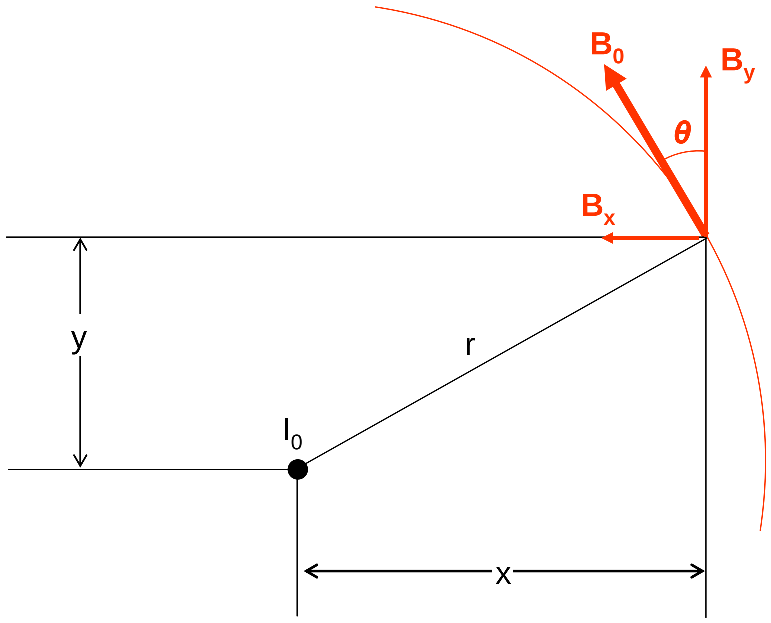
We can calculate r from Pythagoras:

$$r = \left(x^2 + y^2\right)^{1/2}$$

So

$$B = \frac{\mu_0 I_0}{2\pi\left(x^2 + y^2\right)^{1/2}}$$

Next we want to know the vertical and horizontal components of B:



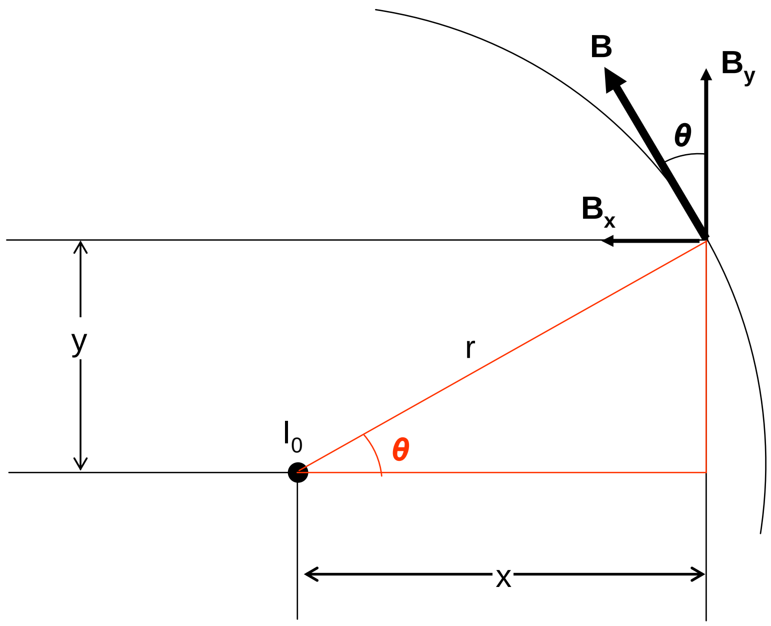
Fairly obviously

$$B_x = -B_0 \sin(\theta)$$

$$B_y = B_0 \cos(\theta)$$

(Note: with the coordinates set up the way we have, and the current assumed to be positive coming out of the page, a minus sign appears in the B_x term. With different coordinate conventions, the minus sign might be different, but the final answer would not change as long as we applied our chosen convention consistently.)

But we can calculate sin and cos of θ:



$$\cos(\theta) = \frac{x}{r} = \frac{x}{\left(x^2 + y^2\right)^{1/2}}$$

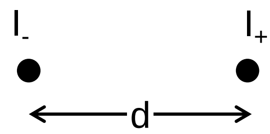
$$\sin(\theta) = \frac{y}{r} = \frac{y}{\left(x^2 + y^2\right)^{1/2}}$$

Hence

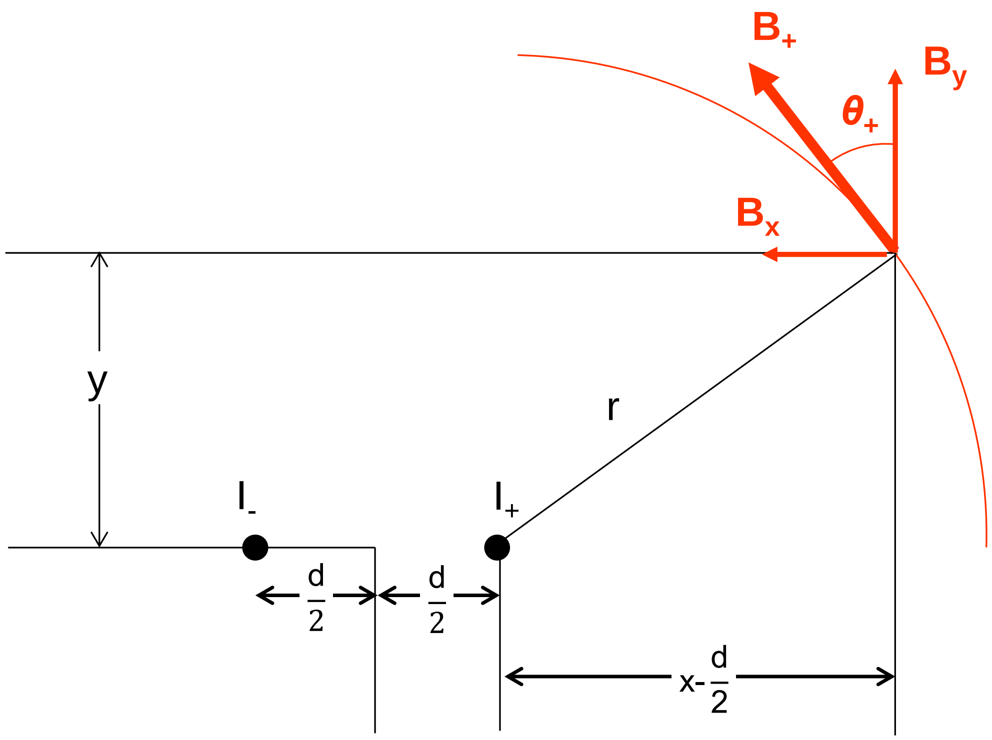
$$B_x = -\frac{\mu_0 I_0}{2\pi\left(x^2 + y^2\right)^{1/2}}\sin(\theta) = -\frac{\mu_0 I_0}{2\pi\left(x^2 + y^2\right)^{1/2}}\frac{y}{\left(x^2 + y^2\right)^{1/2}} = -\frac{\mu_0 I_0 y}{2\pi\left(x^2 + y^2\right)}$$

$$B_y = \frac{\mu_0 I_0}{2\pi\left(x^2 + y^2\right)^{1/2}}\cos(\theta) = \frac{\mu_0 I_0}{2\pi\left(x^2 + y^2\right)^{1/2}}\frac{x}{\left(x^2 + y^2\right)^{1/2}} = \frac{\mu_0 I_0 x}{2\pi\left(x^2 + y^2\right)}$$

Now we have calculated the field from a single current. We now need to consider the two currents that make the circuit together. We choose to consider a flat circuit with conductors spaced d apart in the x direction here, but the principles apply to any geometry.



Consider first the I_+ current and the field it produces:



The geometry is just the same as for the I_0 current but with x replaced by x-d/2. So we can calculate the field components in exactly the same way, replacing x with x-d/2:

$$B_x = -\frac{\mu_0 I_+ y}{2\pi\left(\left(x - {}^{d}/_{2}\right)^2 + y^2\right)}$$

$$B_y = \frac{\mu_0 I_+\left(x - {}^{d}/_{2}\right)}{2\pi\left(\left(x - {}^{d}/_{2}\right)^2 + y^2\right)}$$

The *I*- terms are just the same but with x+d/2 instead of x-d/2, and an extra minus sign because the current is in the opposite direction. (If we had a vertical array instead of a horizontal array, the “d/2” terms would appear with the y terms instead of the x, and if the array was more complicated, there would be modifications to both the x and y terms.) So now we have the vertical and horizontal components for both currents:

	I_-	I_+
B_x	$B_x = \frac{\mu_0 I_+ y}{2\pi\left(\left(x + {}^{d}/_{2}\right)^2 + y^2\right)}$	$B_x = -\frac{\mu_0 I_+ y}{2\pi\left(\left(x - {}^{d}/_{2}\right)^2 + y^2\right)}$
B_y	$B_y = -\frac{\mu_0 I_+\left(x + {}^{d}/_{2}\right)}{2\pi\left(\left(x + {}^{d}/_{2}\right)^2 + y^2\right)}$	$B_y = \frac{\mu_0 I_+\left(x - {}^{d}/_{2}\right)}{2\pi\left(\left(x - {}^{d}/_{2}\right)^2 + y^2\right)}$

To get the total field in each direction, we simply add these up:

$$\begin{aligned} B_x &= \frac{\mu_0 I y}{2\pi\left(\left(x + {}^{d}/_{2}\right)^2 + y^2\right)} - \frac{\mu_0 I y}{2\pi\left(\left(x - {}^{d}/_{2}\right)^2 + y^2\right)} \\ &= \frac{\mu_0 I y}{2\pi}\left(\frac{1}{\left(x + {}^{d}/_{2}\right)^2 + y^2} - \frac{1}{\left(x - {}^{d}/_{2}\right)^2 + y^2}\right) \end{aligned}$$

$$\begin{aligned} B_y &= \frac{\mu_0 I\left(x - {}^{d}/_{2}\right)}{2\pi\left(\left(x - {}^{d}/_{2}\right)^2 + y^2\right)} - \frac{\mu_0 I\left(x + {}^{d}/_{2}\right)}{2\pi\left(\left(x + {}^{d}/_{2}\right)^2 + y^2\right)} \\ &= \frac{\mu_0 I}{2\pi}\left(\frac{x - {}^{d}/_{2}}{\left(x - {}^{d}/_{2}\right)^2 + y^2} - \frac{x + {}^{d}/_{2}}{\left(x + {}^{d}/_{2}\right)^2 + y^2}\right) \end{aligned}$$

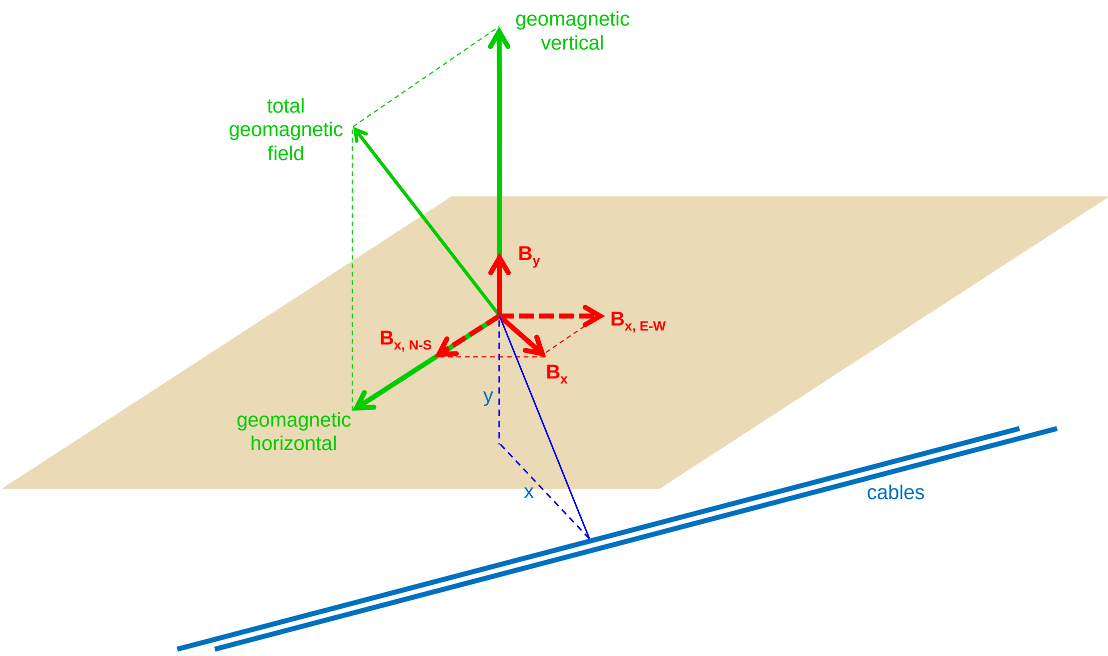
We could simplify these equations further, but usually we would be calculating the answers numerically, so there is little point.

We can now do one of two things, depending on what we want to know.

If we want to know the total field from the cable, we just calculate it as root-sum-of-squares:

$$B_{total} = \sqrt{B_x^2 + B_y^2}$$

Often, however, we are interested in not just the field from the cable alone, but in how it adds to the earth's DC field, so as to give the total DC field (earth plus cable) at a given point. This requires us to think slightly more in three dimensions.



The geomagnetic field is at an angle to the vertical. We can resolve it into two components: the vertical component, and the horizontal component, which, by definition, is north-south with no east-west component.

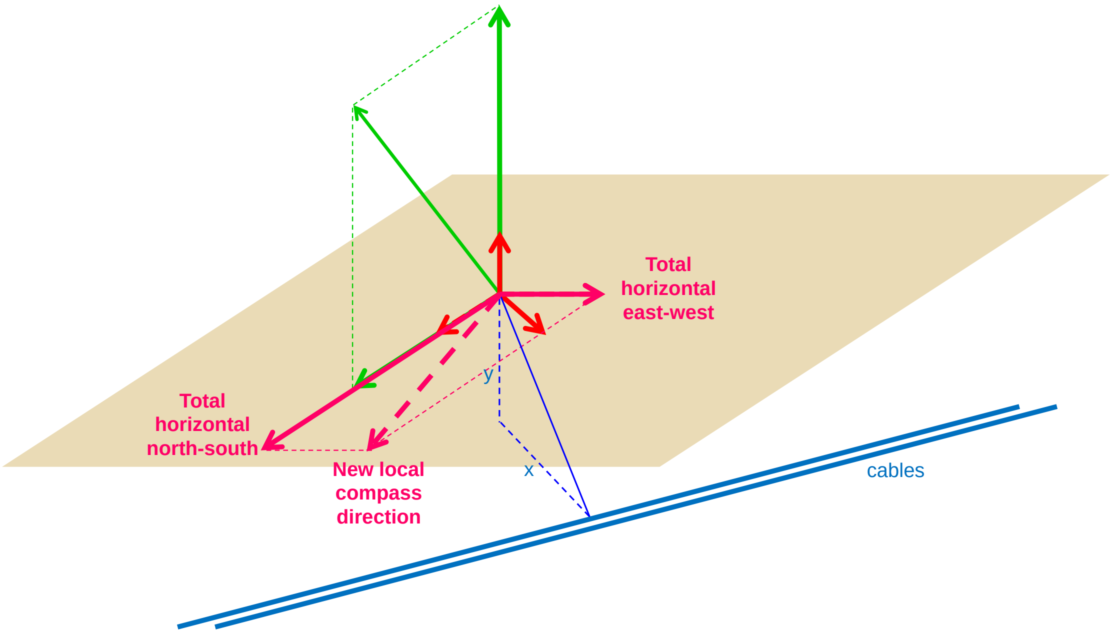
For our cable, a distance y below the ground and x off to the side, we already know how to calculate the vertical and horizontal components B_y and B_x. The total vertical field is easy: it's just the vertical component of the geomagnetic field added to the vertical field B_y from the cable (making sure we have a consistent sign convention as to which direction we count as a positive field). For the horizontal field, it's slightly more complicated. The cable will usually have some arbitrary direction, neither north-south nor east-west. So the horizontal field from the cable, B_x, is also in an arbitrary direction. We need to resolve it into the east-west and north-south components, using the bearing of the cable and the appropriate sine or cos function. Then we can add the north-south component of the cable field, $B_{x,\,N\text{-}S}$, to the horizontal component of the geomagnetic field. For the east-west component, that comes only from the cable, $B_{x,\,E\text{-}W}$, with no geomagnetic component.

Thus we get:

Vertical	geomagnetic vertical + B_y
Horizontal, north-south	geomagnetic horizontal + $B_{x,\,N\text{-}S}$
Horizontal, east-west	$B_{x,\,E\text{-}W}$

Once we've calculated these three components – vertical, horizontal north-south, and horizontal east-west – we can add them by root-sum-of-squares to get the total DC field.

We can also calculate the direction a compass would point along. In the absence of the cable, it just points along the horizontal component of the geomagnetic field, north-south. In the presence of the cable, it points along the new local horizontal field, that is, the resultant of the horizontal north-south and horizontal east-west fields:



More complicated situations:

- If you have more than one circuit, work out all the separate components, sum all the B_x components from each separate conductor, likewise sum all the B_y components, then take the root-sum-of-squares total as before.
- If the currents aren't infinitely long or straight, break them into short straight sections and calculate the components for each of these separately (you can no longer use Ampere's law as it stands, obviously, you have to reduce it by the sin of the angle subtended by the ends of the section at the point of interest).