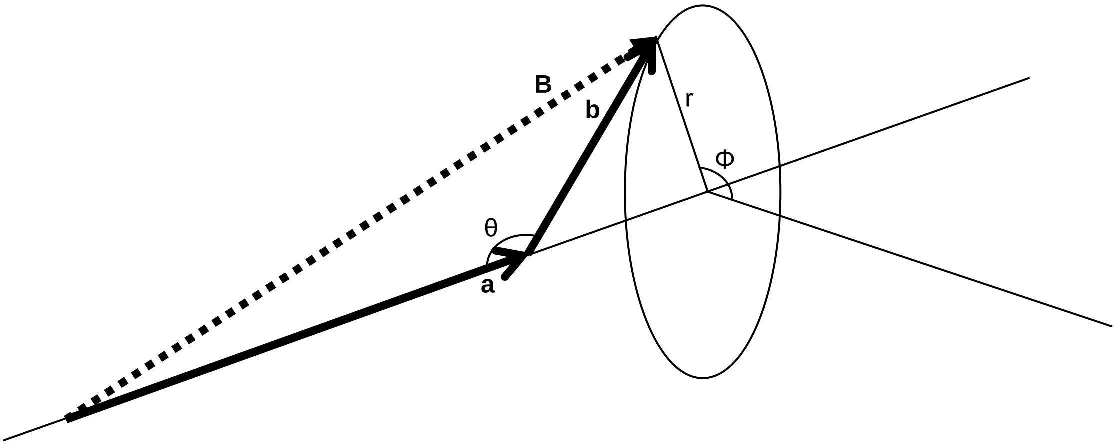
Addition of two magnetic-field vectors with random orientation

Consider two vectors a and b at an angle θ producing a resultant B:



If we know θ, we can calculate B exactly. We address here the problem when we do not know θ. What is the best estimate of B? Or alternatively, if the two vectors have a random and therefore changing relative orientation, what is the average of B averaged over time?

From the cosine rule:

$$B^2 = a^2 + b^2 - 2ab\cos(\theta)$$

We can see that B^2 is symmetrically distributed about a^2+b^2, ie about θ=90°. So

$$B_{median} = \sqrt{a^2 + b^2}$$

and

$$B_{rms} = \left\{\frac{1}{\pi}\int_0^\pi B^2 d\theta\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{1}{\pi}\int_0^\pi \left(a^2 + b^2 - 2ab\cos(\theta)\right)d\theta\right\}^{\frac{1}{2}}$$

$$= \sqrt{a^2 + b^2}$$

In other words, because the distribution is symmetrical about θ=90°, both the median and the rms are given by the value of B when θ=90°. However, B itself is not symmetrically distributed about θ=90°, because taking the square root weights the values below θ=90° more highly than the values above θ=90°. We can calculate B_{mean} from

$$B_{mean} = \int_{\theta,\phi} B(\theta,\phi)\, f(\theta,\phi)$$

where f(θ,Φ) is the probability density function of B.

To calculate f(θ,Φ), note that the set of all possible values of B is given by the surface of the sphere radius b centred on the end of a, area $4\pi b^2$. In other words, we have integrated over all possible values of θ and Φ. We assume all points on this surface are equally likely; that is what we mean by the two vectors having “random” orientation. B constitutes an area of the surface

$$(bd\theta).(rd\phi)$$
$$= bd\theta . b\sin(\theta)d\phi$$

Therefore

$$f(\theta) = \frac{b^2 \sin(\theta)d\theta d\phi}{4\pi b^2}$$
$$= \frac{\sin(\theta)d\theta d\phi}{4\pi}$$

and

$$B_{mean} = \int_0^{\pi} B(\theta)\, f(\theta)$$
$$= \frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(a^2 + b^2 - 2ab\cos(\theta)\right)^{1/2}\sin(\theta)d\theta d\phi$$
$$= \frac{1}{4\pi}2\pi\frac{2}{3}\frac{1}{2ab}\left[\left(a^2 + b^2 - 2ab\cos(\theta)\right)^{3/2}\right]_{\theta=0}^{\pi}$$
$$= \frac{1}{6}\frac{\left(a^2 + b^2 + 2ab\right)^{3/2} - \left(a^2 + b^2 - 2ab\right)^{3/2}}{ab}$$
$$= \frac{1}{6}\frac{(a+b)^3 - (a-b)^3}{ab} \quad or \quad = \frac{1}{6}\frac{(a+b)^3 - (b-a)^3}{ab}$$

Note that in the last line when we simplify the terms we are implicitly taking a square root so we have to decide on which sign to take. Trial and error shows that taking (a-b) in the second term gives sensible answers for a>b (ie answers that are correct in the limiting cases and plausible for the intermediate cases), and taking (b-a) gives the right answer for b>a. Proceeding for the case a>b:

$$B_{mean} = \frac{1}{6}\frac{(a^3 + 3a^2b + 3ab^2 + b^3) - (a^3 - 3a^2b + 3ab^2 - b^3)}{ab}$$
$$= \frac{1}{6}\frac{6a^2b + 2b^3}{ab}$$
$$= a\left(1 + \frac{1}{3}\left(\frac{b}{a}\right)^2\right)$$

If we compare B_{rms} and B_{mean} for some simple special cases, in each case setting a=1:

For a=b:

$$B_{rms} = \sqrt{2} \approx 1.41$$
$$B_{mean} = \frac{4}{3} \approx 1.33$$

For a=2b:

$$B_{rms} = \sqrt{5/4} \approx 1.12$$

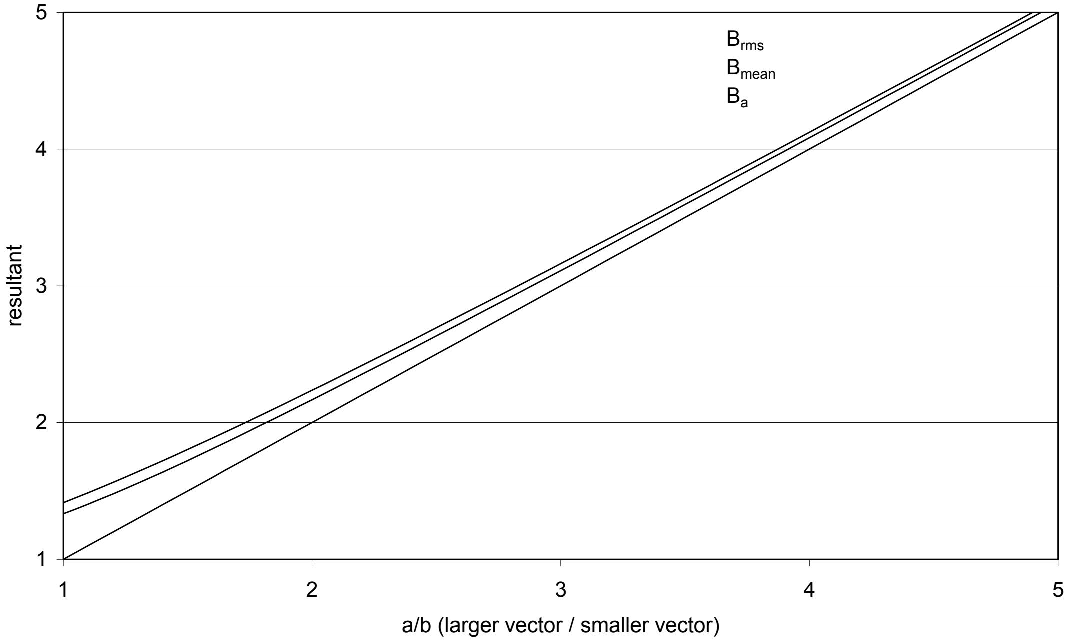
$$B_{mean} = \frac{13}{12} \approx 1.08$$

For the limit of b/a →0 (expanding B_{rms} as a McLaurin series)

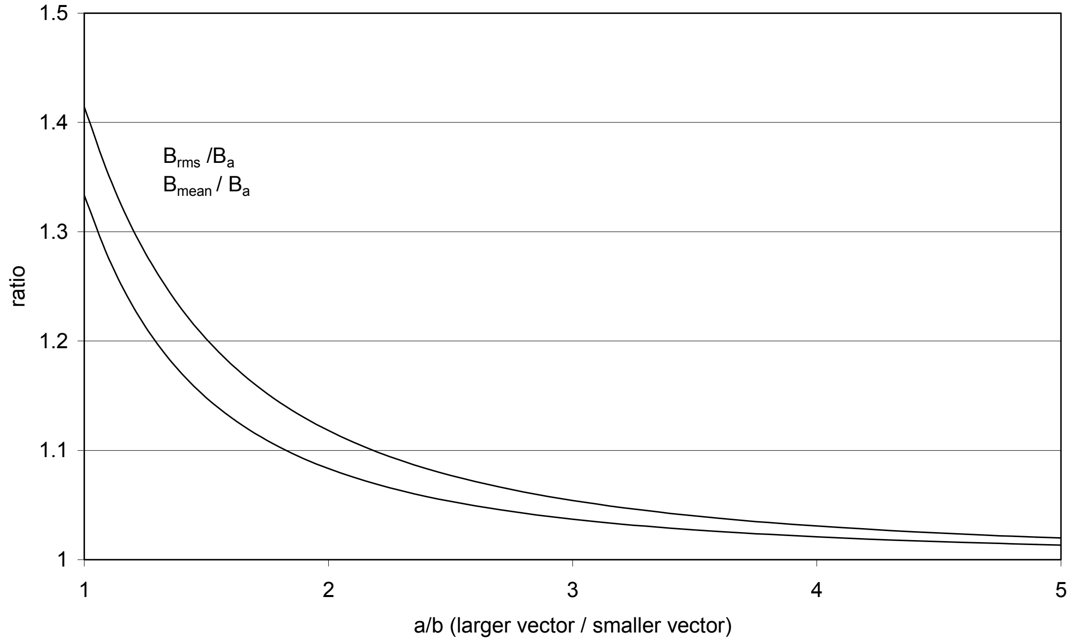
$$B_{rms} = 1 + \frac{b^2}{2}$$

$$B_{mean} = 1 + \frac{b^2}{3}$$

The following graph shows B_{rms} and B_{mean} as a function of a/b:



And this graph shows the same data expressed as the ratio B_{rms}/B_a and B_{mean}/B_a:



Comment:

Intuition might suggest that random vectors averaged over time combine as root-sum-of-squares on the basis that the “average” orientation of the vectors is at right angles. We now see that this is exactly true for the rms and for the median but not for the mean; the mean is numerically slightly lower than the rms. This means the consequence that we usually state as stemming from the random vector addition – that the larger vector always tends to dominate – is even more true than the intuitive result suggests.